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The problem of interactions in de Sitter invariant theories

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Abstract. The structure of representations describing systems of free particles in the theory with the invariance group SO(1, 4) is investigated. The property of the particles to be free means as usual that the representation describing a many-particle system is the tensor product of the corresponding single-particle representations (i.e. no interaction is introduced). It is shown that the mass operator contains only continuous spectrum in the interval $(-\infty, \infty)$ and such representations are unitarily equivalent to ones describing interactions (gravitational, electromagnetic etc). This means that there are no bound states in the theory and the Hilbert space of the many-particle system contains a subspace of states with the following property: the action of free representation operators on these states is manifested in the form of different interactions. Possible consequences of the results are discussed.

1. Introduction

Existing quantum theories are usually based on the following procedure: the Lagrangian of the system under consideration is written as $L = L_{\mathrm{m}} + L_{\mathrm{g}} + L_{\mathrm{int}}$ where L_{m} is the Lagrangian of 'matter', L_{g} is the Lagrangian of gauge fields and L_{int} is the interaction Lagrangian. The symmetry conditions do not define L_{int} uniquely since at least the interaction constant is arbitrary. Nevertheless such an approach has turned out to be highly successful in quantum electrodynamics, electroweak theory and quantum chromodynamics. At the same time the difficulties in constructing quantum gravity have not been overcome.

In the literature lots of possibilities have been considered when interactions are not directly introduced but manifest themselves as a consequence of a non-trivial structure of spacetime. For example, in Kaluza–Klein and superstring theories interactions arise as a consequence of extra dimensions of spacetime while in chiral theories they arise as a consequence of the fact that the Lagrangian is defined on a nonlinear manifold. On the other hand, as it has become clear already in 30th, there is no operator corresponding to spacetime and therefore the notion of the latter on a quantum level is not quite clear.

The problem arises whether the possibility exists that quantum theory is fully defined by the choice of the symmetry group (i.e. there is no need to explicitly introduce Lagrangians, interactions and spacetime) and such a theory can effectively describe the existing interactions.

In this paper we consider a model which in our opinion can shed light on this problem. We choose the de Sitter group SO(1, 4) (more precisely its covering group $\overline{\mathrm{SO}(1,4)}$) as the symmetry group. We require as usual that the elementary particles are described by unitary irreducible representations (UIRs) of this group and different realizations of

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such representations are described in section 2. We then assume that the representation describing the many-particle system is the tensor product of the corresponding single-particle representations. According to the usual philosophy this means that the particles are free and no interaction is introduced. In section 3 we explicitly calculate the free many-particle mass operator and show that the spectrum of this operator contains the whole interval $(-\infty, \infty)$. As shown in section 4, such an operator is unitarily equivalent to the mass operator containing interactions (gravitational, strong, electromagnetic etc). Finally, section 5 is a discussion.

It is worth noting that the unusual properties of SO(1, 4)-invariant theories considered in this paper are specific only for these theories while SO(2, 3)-invariant theories have many common features with Poincaré-invariant ones (the mass of the elementary particle coincides with the minimal value of its energy, the mass of the two-particle system has minimal value $m_1 + m_2$ etc [1]).

2. Realizations of single-particle representations of the SO(1, 4) group

The de Sitter group SO(1, 4) is the symmetry group of the four-dimensional manifold which can be described as follows. If $(x_0, x_1, x_2, x_3, x_4)$ are the coordinates in five-dimensional space, the manifold is the set of points satisfying the condition

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = -R^2 \tag{1}$$

where $R > 0$ is some parameter. Elements of a map of the point $(0, 0, 0, 0, R)$ (or $(0, 0, 0, 0, -R)$) can be parametrized by the coordinates (x_0, x_1, x_2, x_3). If R is very large then such a map proceeds to Minkowski space and the action of the de Sitter group on this map—to the action of the Poincaré group. The quantity R^2 is often written as $R^2 = 3/\Lambda$ where Λ is the cosmological constant. Existing astronomical data show that Λ is very small and the usual estimates based on popular cosmological models give $R > 10^{26}$ cm. On the other hand, in models based on de Sitter cosmology, R_0 is related to the Hubble constant H as $R = 1/H$ and in this case R is of the order 10^{27} cm [2].

The representation generators of the SO(1, 4) group L^{ab} ($a, b = 0, 1, 2, 3, 4$, $L^{ab} = -L^{ba}$) should satisfy the commutation relations

$$[L^{ab}, L^{cd}] = -\imath(\eta^{ac}L^{bd} + \eta^{bd}L^{as} - \eta^{ad}L^{bc} - \eta^{bc}L^{ad}) \tag{2}$$

where η^{ab} is the diagonal metric tensor such that $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = -\eta^{44} = 1$.

In conventional quantum theory elementary particles are described by UIRs of the symmetry group or its Lie algebra. If one assumes that the role of the symmetry group is played by the Poincaré group, then the representations are described by ten generators—six generators of the Lorentz group and the four-momentum operator. In units $c = \hbar = 1$ the former are dimensionless while the latter has the dimension (length)$^{-1}$. If, however, the symmetry group is the de Sitter group SO(1, 4), then all the generators in units $c = \hbar = 1$ are dimensionless. There exists wide literature devoted to the UIRs of this group (see, e.g., [3–9]). In particular the first complete mathematical classification of UIRs has been given in [3], the three realizations of the UIRs discussed later were first considered in [4] and their physical context was discussed in [5]. The reader can explicitly verify that for all realizations the generators indeed satisfy equation (2).

If s is the spin of the particle under consideration, then we use $\|\ldots\|$ to denote the norm in the space of the UIR of group SU(2) with spin s. Let $v = (v_0 = (1 + \mathbf{v}^2)^{1/2}, \mathbf{v})$ be the element of the Lorentz hyperboloid of four velocities and dv be the Lorentz invariant volume element on this hyperboloid. Then, one can realize the UIR under consideration in the space of functions $\{f_1(v), f_2(v)\}$ on two Lorentz hyperboloids with the range in the space of the UIR of group SU(2) with spin s such that

$$\int (\|f_1(v)\|^2 + \|f_2(v)\|^2)\,\mathrm{d}v < \infty. \tag{3}$$

The explicit calculation shows that the action of the generators on $f_1(v)$ has the form

$$\begin{aligned}
&\boldsymbol{M} = l(\boldsymbol{v}) + \boldsymbol{s} \qquad \boldsymbol{N} = -\iota v_0 \frac{\partial}{\partial \boldsymbol{v}} + \frac{\boldsymbol{s}\times\boldsymbol{v}}{v_0+1}\\
&\boldsymbol{B} = \mu\boldsymbol{v} + \iota\left(\frac{\partial}{\partial\boldsymbol{v}} + \boldsymbol{v}\left(\boldsymbol{v}\frac{\partial}{\partial\boldsymbol{v}}\right) + \frac{3}{2}\boldsymbol{v}\right) + \frac{\boldsymbol{s}\times\boldsymbol{v}}{v_0+1}\\
&L_{04} = \mu v_0 + \iota v_0\left(\boldsymbol{v}\frac{\partial}{\partial\boldsymbol{v}} + \frac{3}{2}\right)
\end{aligned} \tag{4}$$

where $\boldsymbol{M} = \{L^{23}, L^{31}, L^{12}\}$, $\boldsymbol{N} = \{L^{01}, L^{02}, L^{03}\}$, $\boldsymbol{B} = -\{L^{14}, L^{24}, L^{34}\}$, $\boldsymbol{s}$ is the spin operator and $\boldsymbol{l}(\boldsymbol{v}) = -\iota\boldsymbol{v}\times\partial/\partial\boldsymbol{v}$. The action of L^{ab} on $f_2(v)$ is obtained from equation (4) by the substitution $\mu\to-\mu$.

Such a realization is used to obtain the possible closest analogy between representations of the SO(1, 4) and the Poincaré group. It is easy to see that the operators $\boldsymbol{M}$ and $\boldsymbol{N}$ in equation (4) have the same form as for the standard realization of the single-particle representations of the Poincaré group (see, e.g., [10, 11]) while the contraction of equation (4) into the standard realization of the Poincaré group is accomplished as follows. Denote $m = \mu/R$, $\boldsymbol{P} = \boldsymbol{B}/R$ and $E = L_{04}/R$ and consider the action of the generators on functions which do not depend on R in the usual system of units. Then, as follows from equations (1) and (4), in the limit $R\to\infty$ we obtain the standard representation of the Poincaré group for a particle with mass m, since $\boldsymbol{P} = m\boldsymbol{v}$, $E = mv_0$ (in this case one represents the Poincaré group with negative energy on the second hyperboloid).

Since the representation generators of the SO(1, 4) group are dimensionless (in units $c = \hbar = 1$), any quantal description in SO(1, 4)-invariant theory involves only dimensional quantities. In particular, as seen from equation (4), the quantal description of particles in such a theory does not involve any information about the quantity R (this property is clear from the fact that the elements of the SO(1, 4) group describe only homogeneous transformations of the manifold defined by equation (1)). The latter comes into play only when we wish to interpret the results in terms of quantities used in Poincaré-invariant theory. Therefore, if we assume that de Sitter invariance is fundamental and Poincaré invariance is only approximate, it is reasonable to think that the de Sitter masses μ are fundamental while m and R are not (see also the discussion in [12]).

It is also possible to realize the UIR in the space of functions $\varphi(u)$ on the three-dimensional unit sphere S^3 in four-dimensional space with the range in the space of the UIR of the group SU(2) with spin s such that

$$\int \|\varphi(u)\|^2\,\mathrm{d}u < \infty \tag{5}$$

where $\mathrm{d}u$ is the SO(4)-invariant volume element on S^3. Elements of S^3 can be represented as $u = (\boldsymbol{u}, u_4)$ where $u_4 = \pm(1-\boldsymbol{u}^2)^{1/2}$ for the upper and lower hemispheres, respectively. Then, the explicit calculation shows that for this realization the generators have the form

$$\begin{aligned}
&\boldsymbol{M} = l(\boldsymbol{u}) + \boldsymbol{s} \qquad \boldsymbol{B} = \iota u_4\frac{\partial}{\partial\boldsymbol{u}} - \boldsymbol{s}\\
&\boldsymbol{N} = \iota\left(\frac{\partial}{\partial\boldsymbol{u}} - \boldsymbol{u}\left(\boldsymbol{u}\frac{\partial}{\partial\boldsymbol{u}}\right)\right) - \left(\mu + \frac{3\iota}{2}\right)\boldsymbol{u} + \boldsymbol{u}\times\boldsymbol{s} - u_4\boldsymbol{s}
\end{aligned}$$

$$L_{04} = \left(\mu + \frac{3\iota}{2}\right)u_4 + \iota u_4 \boldsymbol{u}\frac{\partial}{\partial \boldsymbol{u}}. \tag{6}$$

Since equations (3) and (4) on the one hand and equations (5) and (6) on the other are different realizations of one and the same representation, there exists a unitary operator transforming functions $f(v)$ into $\varphi(u)$ and operators (4) into operators (6). For example, in the spinless case

$$\varphi(u) = \exp\left(-\frac{\iota}{2}\mu \log v_0\right)v_0^{3/2} f(v) \tag{7}$$

where $\boldsymbol{u} = -\boldsymbol{v}/v_0$. In view of this relation, the sphere S^3 is usually interpreted in the literature as the velocity space (see, e.g., [7, 8]) but, as argued in [9], there are serious arguments for interpreting S^3 as the coordinate space. Later, we give additional arguments in favour of this point of view.

As follows from equation (1), if x_0 is fixed then the set of points satisfying this relation is the three-dimensional sphere $S^3(R_1)$ with radius $R_1 = (R^2 + x_0^2)^{1/2}$. This sphere is invariant under the action of the SO(4) subgroup of the SO(1, 4) group. The operators $\boldsymbol{B}$ and $\boldsymbol{M}$ are the representation generators of the SO(4) subgroup. We can choose $\boldsymbol{x} = R_1\boldsymbol{u}$ as the coordinates on $S^3(R_1)$. In these coordinates the operators $\boldsymbol{B}$ and $\boldsymbol{M}$ given by equation (6) are the generators of the representation of the group of motions of $S^3(R_1)$ induced from the representation of the SO(3) subgroup with generators $\boldsymbol{s}$.

Consider a vicinity of the south pole of $S^3(R_1)$ such that $|\boldsymbol{x}| \ll R_1$ and assume that $x_0 \ll R$. Then, the generators in equation (6) have the form

$$\boldsymbol{B} = R\boldsymbol{p} \qquad \boldsymbol{M} = \boldsymbol{l}(\boldsymbol{x}) + \boldsymbol{s} \qquad \boldsymbol{N} = -R\boldsymbol{p} \qquad L_{04} = mR \tag{8}$$

where $\boldsymbol{p} = -\iota\partial/\partial\boldsymbol{x}$ and $m = \mu/R$. Therefore, $\boldsymbol{B}/R$ is the de Sitter analogue of the momentum operator, but L_{04}/R in this realization is not the de Sitter analogue of the energy operator.

The reason for such a situation is as follows. In Poincaré-invariant theories one can consider wavefunctions defined on the conventional three-dimensional space R^3. The operators $\boldsymbol{M}$ and $\boldsymbol{P}$ are the representation generators of the group of motions of R^3. From the remaining generators, E and $\boldsymbol{N}$, only E commutes with $\boldsymbol{M}$ and $\boldsymbol{P}$. For this reason E can be chosen as the operator responsible for the evolution of the system under consideration while stationary states are eigenstates of E. In the SO(1, 4) case one can consider wavefunctions defined on $S^3(R_1)$ at different values of x_0. However, none of the generators L_{04}, $\boldsymbol{N}$ commutes with all the operators $\boldsymbol{M}$ and $\boldsymbol{B}$. At the same time the operator $E_{\mathrm{dS}} = (L_{04}^2 + \boldsymbol{N}^2)^{1/2}$ satisfies this property. Hence E_{dS} can be treated as the operator responsible for the evolution. At conditions described by equation (8), $E_{\mathrm{dS}} = R(m^2 + \boldsymbol{p}^2)^{1/2}$ and therefore, E_{dS} can be considered as the de Sitter analogue of the energy operator.

The inconvenience of working with $\boldsymbol{B}$ as the de Sitter analogue of the momentum operator is that different components of $\boldsymbol{B}$ commute with each other only when $R \to \infty$. We can define the operators $\boldsymbol{Q}_+ = (L_{1+}, L_{2+}, L_{3+})$ and $\boldsymbol{Q}_- = (L_{1-}, L_{2-}, L_{3-})$, where the $\pm$ components of five vectors are defined as $x_\pm = x_4 \pm x_0$. Then, as follows from equation (2), different components of $\boldsymbol{Q}_+$ commute with each other and the same is valid for $\boldsymbol{Q}_-$.

It is easy to see that $2\boldsymbol{u}/(1 - u_4)$ is the stereographic projection of the point $(\boldsymbol{u}, u_4) \in S^3$ onto three-dimensional space. Now we use $\boldsymbol{x}$ to denote the quantity $\boldsymbol{x} = 2R\boldsymbol{u}/(1 - u_4)$. In the space of functions $\varphi(\boldsymbol{x})$ on R^3 with the range in the space of the UIR of the group SU(2) with spin s and such that

$$\int \|\varphi(\boldsymbol{x})\|^2 \,\mathrm{d}^3\boldsymbol{x} < \infty \tag{9}$$

the generators of the UIR of the SO(1, 4) group can be realized as

$$\boldsymbol{M} = \boldsymbol{l}(\boldsymbol{x}) + \boldsymbol{s} \qquad L_{+-} = -2(\mu + \boldsymbol{x}\boldsymbol{p}) + 3\imath \qquad \boldsymbol{Q}_+ = -2R\boldsymbol{p}$$
$$\boldsymbol{Q}_- = \frac{1}{2R}(-2\mu\boldsymbol{x} + \boldsymbol{x}^2\boldsymbol{p} - 2\boldsymbol{x}(\boldsymbol{x}\boldsymbol{p}) + 3\imath\boldsymbol{x} + 2(\boldsymbol{s}\times\boldsymbol{x})). \tag{10}$$

The UIR realized by equations (4), (6) or (10) belongs to the so-called principal series. It can be characterized by the condition that $\mu^2 \geqslant 0$, i.e. μ is real. In contrast with UIRs of the Poincaré group, the sign of μ does not make it possible to distinguish UIRs describing particles and antiparticles (see, e.g., the discussion in [7]) and the UIRs with μ and $-\mu$ are unitarily equivalent.

The Casimir operator of second order can be written as

$$I_2 = -\sum_{ab} L_{ab}L^{ab} = \frac{1}{2}(L_{+-})^2 - (\boldsymbol{Q}_+\boldsymbol{Q}_- + \boldsymbol{Q}_-\boldsymbol{Q}_+) - 2\boldsymbol{M}^2. \tag{11}$$

A direct calculation shows that in the case described by equations (4), (6) and (10)

$$I_2 = 2\left(\mu^2 - \boldsymbol{s}^2 + \frac{9}{4}\right). \tag{12}$$

In Poincaré-invariant theories the spectrum of the mass operator can be defined as the spectrum of the energy operator in the subspace of states with zero total momentum. As follows from equation (10), for UIRs of the SO(1, 4) group corresponding to the principal series, the spectrum of the mass operator can be defined from the condition that the action of L_{+-} on states with zero momentum is equal to $-2\mu + 3\imath$. The presence of $3\imath$ in this expression does not contradict the Hermiticity of L_{+-} since L_{+-} does not commute with $\boldsymbol{Q}_+$.

3. Free many-particle mass operator in the SO(1, 4)-invariant theory

The representation describing a system of N non-interacting particles is constructed as the tensor product of corresponding single-particle representations and the representation generators are equal to the sums of single-particle generators, i.e.

$$L_{ab} = \sum_{n=1}^{N} L_{ab}^{(n)} \tag{13}$$

where $L_{ab}^{(n)}$ are generators for the nth particle. Each generator acts through variables of its 'own' particle, as described in the preceding section and through variables of other particles it acts as the identity operator.

The tensor product of single-particle representations can be decomposed into the direct integral of UIRs and there exists a well elaborated general theory [13]. In the given case, among UIRs there may not only be representations of the principal series but also UIRs of other series.

We first consider the case of two particles 1 and 2. Suppose that the UIRs for them are realized as in equation (4). We introduce conventional masses and momenta $m_j = \mu_j/R$, $\boldsymbol{p}_j = m_j\boldsymbol{v}_j$ $(j = 1, 2)$. We can define the variables describing the system as a whole and the internal variables. The usual non-relativistic variables are

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 \qquad \boldsymbol{k} = \frac{m_2\boldsymbol{p}_1 - m_1\boldsymbol{p}_2}{m_1 + m_2}. \tag{14}$$

Then, in the approximation when particle velocities are very small, it follows from equations (4) and (13) that the two-particle generators have the form

$$\boldsymbol{M} = \boldsymbol{l}(\boldsymbol{P}) + \boldsymbol{S} \qquad \boldsymbol{N} = -\imath(m_1 + m_2)\frac{\partial}{\partial\boldsymbol{P}}$$

$$L_{04} = R(m_1+m_2) + \iota\left(\boldsymbol{k}\frac{\partial}{\partial\boldsymbol{k}} + \frac{3}{2}\right) + \iota\left(\boldsymbol{P}\frac{\partial}{\partial\boldsymbol{P}} + \frac{3}{2}\right)$$
$$\boldsymbol{B} = R\boldsymbol{P} + \iota(m_1+m_2)\frac{\partial}{\partial\boldsymbol{P}} \tag{15}$$

where $\boldsymbol{S} = \boldsymbol{l}(\boldsymbol{k}) + \boldsymbol{s}_1 + \boldsymbol{s}_2$. Comparison of equations (4) and (15) for $\boldsymbol{M}$ shows that $\boldsymbol{S}$ plays the role of the spin operator for the system as a whole (in full analogy with conventional quantum mechanics).

By analogy with equation (12) we can define the mass operator M_{dS} for the system as a whole. Namely, if I_2 is the Casimir operator for the system as a whole defined by this equation, then

$$I_2 = 2(M_{\mathrm{dS}}^2 - \boldsymbol{S}^2 + \tfrac{9}{4}). \tag{16}$$

In turn, the conventional mass operator M can be defined as M_{dS}/R.

As follows from equations (15) and (16), for slow particles of first order in $1/R$

$$M = m_1 + m_2 + \frac{\iota}{R}\left(\boldsymbol{k}\frac{\partial}{\partial\boldsymbol{k}} + \frac{3}{2}\right). \tag{17}$$

We shall see later that this expression is correct for any velocities and of any order in $1/R$ if only the representations of the principal series are taken into account.

Equation (17) means that for very slow particles the de Sitter correction to the classical non-relativistic Hamiltonian is equal to $\Delta H_{\mathrm{nr}} = (\boldsymbol{k}\boldsymbol{r})/R$ where $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ is the vector of the relative distance between the particles (this quantity is conjugated with $\boldsymbol{k}$). As follows from the classical equations of motion, $\ddot{\boldsymbol{r}} = \boldsymbol{r}/R^2$. Therefore, the correction corresponds to the well known fact that in classical SO(1, 4)-invariant theory there exists antigravity and the force of (cosmological) repulsion between particles is proportional to the distance between them. It is also interesting to note that the de Sitter antigravity is in some sense even more universal than usual gravity: the force of repulsion does not depend on the parameters characterizing the particles (even on their masses).

Now we again consider the case of two particles but suppose that the UIRs for them are realized as in equation (10). We introduce the standard non-relativistic variables

$$\boldsymbol{X} = \frac{m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2}{m_1+m_2} = \frac{\mu_1\boldsymbol{x}_1 + \mu_2\boldsymbol{x}_2}{\mu_1+\mu_2} \qquad \boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2. \tag{18}$$

Then, a direct calculation of the two-particle generators gives

$$\begin{aligned}
L_{+-} &= -2(\mu_1+\mu_2+\boldsymbol{X}\boldsymbol{P}) + 2\iota\left(\boldsymbol{r}\frac{\partial}{\partial\boldsymbol{r}} + 3\right)\\
\boldsymbol{Q}_+ &= -2R\boldsymbol{P} \qquad \boldsymbol{M} = \boldsymbol{l}(\boldsymbol{X}) + \boldsymbol{S}\\
\boldsymbol{Q}_- &= -(m_1+m_2)\boldsymbol{X} + \frac{1}{2R}\Bigg(\boldsymbol{X}^2\boldsymbol{P} + \frac{m_1m_2}{(m_1+m_2)^2}\boldsymbol{r}^2\boldsymbol{P}\\
&\quad -2\iota\left(\boldsymbol{r}\boldsymbol{X}\frac{\partial}{\partial\boldsymbol{r}} - \iota\frac{m_2-m_1}{m_1+m_2}\boldsymbol{r}^2\frac{\partial}{\partial\boldsymbol{r}} - 2\boldsymbol{X}(\boldsymbol{X}\boldsymbol{P}) - \frac{2m_1m_2}{(m_1+m_2)^2}\boldsymbol{r}(\boldsymbol{r}\boldsymbol{P})\right.\\
&\quad +2\iota\boldsymbol{X}\left(\boldsymbol{r}\frac{\partial}{\partial\boldsymbol{r}}\right) + 2\iota\boldsymbol{r}\left(\boldsymbol{X}\frac{\partial}{\partial\boldsymbol{r}}\right) + \frac{2\iota(m_2-m_1)}{m_1+m_2}\boldsymbol{r}\left(\boldsymbol{r}\frac{\partial}{\partial\boldsymbol{r}}\right) + 6\iota\boldsymbol{X}\\
&\quad \left.+\frac{3\iota(m_2-m_1)}{m_1+m_2}\boldsymbol{r}\right) + \frac{1}{R}\left((\boldsymbol{s}_1+\boldsymbol{s}_2)\times\boldsymbol{X} + \frac{m_2\boldsymbol{s}_1 - m_1\boldsymbol{s}_2}{m_1+m_2}\times\boldsymbol{r}\right)
\end{aligned} \tag{19}$$

where $\boldsymbol{S} = \boldsymbol{l}(\boldsymbol{r}) + \boldsymbol{s}_1 + \boldsymbol{s}_2$ and $\boldsymbol{P} = -\iota\partial/\partial\boldsymbol{X}$.

A direct calculation shows that, as a consequence of equations (11), (16) and (19),

$$
\begin{aligned}
M_{\mathrm{dS}}^2 = {} & \left(\mu_1 + \mu_2 - \iota\left(\boldsymbol{r}\frac{\partial}{\partial \boldsymbol{r}} + \frac{3}{2}\right)\right)^2 + \frac{\mu_1\mu_2}{(\mu_1+\mu_2)^2}\left(\boldsymbol{r}^2\boldsymbol{P}^2 - 2(\boldsymbol{r}\boldsymbol{P})^2\right) \\
& + \frac{\iota(\mu_2-\mu_1)}{\mu_2+\mu_1}\left(2(\boldsymbol{r}\boldsymbol{P})\left(\boldsymbol{r}\frac{\partial}{\partial \boldsymbol{r}}\right) + 3\boldsymbol{r}\boldsymbol{P} - \boldsymbol{r}^2\left(\boldsymbol{P}\frac{\partial}{\partial \boldsymbol{r}}\right)\right) \\
& + \frac{4(\mu_2\boldsymbol{s}_1 - \mu_1\boldsymbol{s}_2)(\boldsymbol{r}\times\boldsymbol{P})}{\mu_1+\mu_2}
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
M^2 = {} & \left(m_1 + m_2 - \frac{\iota}{R}\left(\boldsymbol{r}\frac{\partial}{\partial \boldsymbol{r}} + \frac{3}{2}\right)\right)^2 + \frac{m_1 m_2}{R^2(m_1+m_2)^2}\left(\boldsymbol{r}^2\boldsymbol{P}^2 - 2(\boldsymbol{r}\boldsymbol{P})^2\right) \\
& + \frac{\iota(m_2-m_1)}{R^2(m_2+m_1)}\left(2(\boldsymbol{r}\boldsymbol{P})\left(\boldsymbol{r}\frac{\partial}{\partial \boldsymbol{r}}\right) + 3\boldsymbol{r}\boldsymbol{P} - \boldsymbol{r}^2\left(\boldsymbol{P}\frac{\partial}{\partial \boldsymbol{r}}\right)\right) \\
& + \frac{4(m_2\boldsymbol{s}_1 - m_1\boldsymbol{s}_2)(\boldsymbol{r}\times\boldsymbol{P})}{R^2(m_1+m_2)}.
\end{aligned} \tag{21}
$$

The expressions for both M_{dS} and M have been explicitly written down in order to stress that if M_{dS} is expressed in terms of de Sitter masses then it does not depend on R. Such a dependence arises only when one considers the conventional mass operator in terms of conventional masses (see the discussion in the preceding section).

As follows from equation (21), the decomposition of the tensor product of two UIRs belonging to the principal series contains not only UIRs belonging to this series. Indeed, the spectrum of the operator M^2 is not positive definite. This is clear, for example, from the fact that for very large values of $|\boldsymbol{P}|$ and values of $\boldsymbol{r}$ collinear with $\boldsymbol{P}$, M^2 becomes negative. However, if R is very large then such values of $|\boldsymbol{P}|$ are practically impossible. It is obvious from equation (21) that for realistic values of $|\boldsymbol{P}|$ the contribution to M^2 of the UIRs not belonging to the principal series is a small correction of order $1/R^2$.

If only the contribution of UIRs belonging to the principal series is taken into account, then the problem of determining M^2 can be considered as follows. Since the tensor product of two UIRs can be decomposed into the direct integral of UIRs [13] and any UIR of the principal series is unitarily equivalent to equation (10) with some operators $\boldsymbol{s}$ and values of μ, we can conclude that any representation of the SO(1, 4) group containing only UIRs of the principal series is unitarily equivalent to the representation defined by the generators

$$
\begin{aligned}
& \boldsymbol{M} = \boldsymbol{l}(\boldsymbol{X}) + \boldsymbol{S} \qquad L_{+-} = -2(M_{\mathrm{dS}} + \boldsymbol{X}\boldsymbol{P}) + 3\iota \qquad \boldsymbol{Q}_+ = -2R\boldsymbol{P} \\
& \boldsymbol{Q}_- = \frac{1}{2R}(-2M_{\mathrm{dS}}\boldsymbol{X} + \boldsymbol{X}^2\boldsymbol{P} - 2\boldsymbol{X}(\boldsymbol{X}\boldsymbol{P}) + 3\iota\boldsymbol{X} + 2(\boldsymbol{S}\times\boldsymbol{X}))
\end{aligned} \tag{22}
$$

where the generators $\boldsymbol{S}$ and M_{dS} act only through the internal variables of the system under consideration. By analogy with the case of UIRs considered in the preceding section, it is clear that M_{dS} can be determined by considering the action of L_{+-} on the states with $\boldsymbol{P} = 0$: the action of L_{+-} on such states is equal to $2(-M_{\mathrm{dS}} + 3\iota/2)$ (recall that the sign of M_{dS} does not play a role). Therefore, as follows from equation (19), the mass operator in the given case is unitarily equivalent to the operator

$$
M = m_1 + m_2 - \frac{\iota}{R}\left(\boldsymbol{r}\frac{\partial}{\partial \boldsymbol{r}} + \frac{3}{2}\right). \tag{23}
$$

In particular, the positive part of the operator (21) is unitarily equivalent to the square of the operator given by equation (23). This is in agreement with the 'common wisdom' according to which the spectrum of the mass operator is defined by its reduction on the (generalized) subspace of states with $\boldsymbol{P} = 0$.

Comparison of equations (17) and (23) is an additional argument in treating $S^3(R_1)$ as the coordinate space, at least at very small velocities (note that $\boldsymbol{r} = \iota\partial/\partial\boldsymbol{k}$, $\boldsymbol{k} = -\iota\partial/\partial\boldsymbol{r}$). We will use momentum and coordinate representations depending on convenience. In the first case the operator given by (17) acts in the space of functions $f(\boldsymbol{k})$ such that

$$\int |f(\boldsymbol{k})|^2 \, \mathrm{d}^3\boldsymbol{k} < \infty \tag{24}$$

and in the second case—in the space of functions $\varphi(\boldsymbol{r})$ such that

$$\int |\varphi(\boldsymbol{r})|^2 \, \mathrm{d}^3\boldsymbol{r} < \infty \tag{25}$$

(here and henceforth we will consider only the spinless case for simplicity)—the functions $f(\boldsymbol{k})$ and $\varphi(\boldsymbol{r})$ are Fourier transforms of each other.

In spherical coordinates $\boldsymbol{r}\partial/\partial\boldsymbol{r} = r\partial/\partial r$ where $r = |\boldsymbol{r}|$. Therefore, in these coordinates the operator (23) does not act through angular variables. We can consider the action of this operator in the space of functions $\varphi(r)$ such that

$$\int |\varphi(r)|^2 r^2 \, \mathrm{d}r < \infty. \tag{26}$$

The eigenvalue problem

$$(m_1 + m_2)\varphi_\lambda(r) - \frac{\iota}{R}\left(r\frac{\mathrm{d}\varphi_\lambda(r)}{\mathrm{d}r} + \frac{3}{2}\varphi_\lambda(r)\right) = \lambda\varphi_\lambda(r) \tag{27}$$

has the solution

$$\varphi_\lambda(r) = \frac{1}{r}\left(\frac{R}{2\pi r}\right)^{1/2} \exp[\iota R(\lambda - m_1 - m_2)\log r] \tag{28}$$

where we assume that r is given in some dimensional units. Then

$$\int_0^\infty \varphi_\lambda(r)^*\varphi_{\lambda'}(r) r^2 \, \mathrm{d}r = \delta(\lambda - \lambda') \qquad \int_{-\infty}^\infty \varphi_\lambda(r)^*\varphi_\lambda(r') \, \mathrm{d}\lambda = \frac{1}{r^2}\delta(r - r') \tag{29}$$

where * means the complex conjugation. Therefore, the operator given in (23) contains only the continuous spectrum occupying the interval $(-\infty, \infty)$. The same is obviously valid for equation (17). As a consequence, we have the following statements.

Statement 1. *The operators given by equations (17) and (23) are unitarily equivalent to the operator*

$$M_0 = \frac{\iota}{R}\left(\boldsymbol{k}\frac{\partial}{\partial\boldsymbol{k}} + \frac{3}{2}\right) = -\frac{\iota}{R}\left(\boldsymbol{r}\frac{\partial}{\partial\boldsymbol{r}} + \frac{3}{2}\right). \tag{30}$$

When $R \to \infty$ we must have compatibility of these results with the standard results of the Poincaré-invariant theory, according to which the mass operator is given by $M^{\mathrm{P}} = \omega_1(k)+\omega_2(k)$ where $\omega_i(k) = (m_i^2 + k^2)^{1/2}$ $(i = 1, 2)$ and $k = |\boldsymbol{k}|$. We use $g(k^2)$ to denote the function

$$g(k^2) = \sum_{i=1}^{2}\left(\omega_i(k) - m_i - m_i \log\frac{\omega_i(k) + m_i}{2m_i}\right). \tag{31}$$

Then, it is obvious that

$$\begin{aligned}\omega_1(k) + \omega_2(k) + \frac{\iota}{R}\left(\boldsymbol{k}\frac{\partial}{\partial\boldsymbol{k}} + \frac{3}{2}\right)\\ = \exp(\iota Rg(k^2))\left(m_1 + m_2 + \frac{\iota}{R}\left(\boldsymbol{k}\frac{\partial}{\partial\boldsymbol{k}} + \frac{3}{2}\right)\right)\exp(-\iota Rg(k^2)).\end{aligned} \tag{32}$$

We can formulate this result as statement 2.

Statement 2. *The operator*

$$\tilde{M} = \omega_1(k) + \omega_2(k) + \frac{\iota}{R}\left(\boldsymbol{k}\frac{\partial}{\partial \boldsymbol{k}} + \frac{3}{2}\right) \tag{33}$$

is unitarily equivalent to the operator given by equation (17).

If one considers the action of the operator $\tilde{M}$ on functions $f(\boldsymbol{k})$ satisfying the conditions

$$|\partial f(\boldsymbol{k})/\partial \boldsymbol{k}| \ll R|f(\boldsymbol{k})| \tag{34}$$

then the action of $\tilde{M}$ is practically indistinguishable from the action of M^{P}.

For the case of three particles with masses m_1, m_2 and m_3 we can introduce the standard Jacobi variables

$$\boldsymbol{k}_{12} = \frac{m_2\boldsymbol{p}_1 - m_1\boldsymbol{p}_2}{m_1 + m_2} \qquad \boldsymbol{K}_{12} = \frac{m_3(\boldsymbol{p}_1 + \boldsymbol{p}_2) - (m_1 + m_2)\boldsymbol{p}_3}{m_1 + m_2 + m_3}. \tag{35}$$

Then by analogy with this consideration one can show that in the approximation when only the representations containing the UIRs of the principal series are taken into account the three-particle mass operator is unitarily equivalent to

$$\tilde{M}_{123} = (M_{12\mathrm{P}}^2 + K_{12}^2)^{1/2} + \omega_3(K_{12}) + \frac{\iota}{R}\left(\boldsymbol{k}_{12}\frac{\partial}{\partial \boldsymbol{k}_{12}} + \frac{3}{2}\right) + \frac{\iota}{R}\left(\boldsymbol{K}_{12}\frac{\partial}{\partial \boldsymbol{K}_{12}} + \frac{3}{2}\right) \tag{36}$$

where M_{12}^{P} is the mass operator of the system given by equation (12) in the Poincaré-invariant theory. There exists a subspace of functions $f(\boldsymbol{k}_{12}, \boldsymbol{K}_{12})$ with the following property: the action of $\tilde{M}_{123}$ on these functions is practically indistinguishable from the action of the standard mass operator $(M_{12\mathrm{P}}^2+K_{12}^2)^{1/2}+\omega_3(K_{12})$. The functions $f(\boldsymbol{k}_{12}, \boldsymbol{K}_{12})$ should satisfy the property

$$\left|\frac{\partial f(\boldsymbol{k}_{12}, \boldsymbol{K}_{12})}{\partial \boldsymbol{k}_{12}}\right|, \left|\frac{\partial f(\boldsymbol{k}_{12}, \boldsymbol{K}_{12})}{\partial \boldsymbol{K}_{12}}\right| \ll R|f(\boldsymbol{k}_{12}, \boldsymbol{K}_{12})|. \tag{37}$$

It is also clear that these results can be generalized to the case when N is arbitrary.

4. Unitary equivalence of free and interacting representations in the SO(1, 4)-invariant theory

If the particles in the system under consideration interact with each other then the representation generators describing this system are interaction-dependent but they should satisfy the commutation relations of (2). By analogy with the procedure proposed by Bakamdjian and Thomas in Poincaré-invariant theories [14], we can introduce the interaction by replacing the free mass operator M_{dS} in equation (22) by an interacting mass operator $\hat{M}_{\mathrm{dS}}$. Then the relations of (22) will obviously be satisfied if $\hat{M}_{\mathrm{dS}}$ acts only through the internal variables and commutes with $\boldsymbol{S}$.

In the general case the spin and momentum operators in equation (22) can be interaction-dependent too but, by analogy with Poincaré-invariant theories (see, e.g., [11]) it is natural to assume that they have the same spectrum as the corresponding free operators. Therefore, one can eliminate the interaction dependence of the spin and momentum operators by using a proper unitary transformation. In Poincaré-invariant theories the corresponding unitary operators are known as Sokolov packing operators (see, e.g., [11, 15–19]).

In this paper we will consider the representation generators in Bakamdjian–Thomas (BT) form, but this discussion gives us grounds to think that if only the contribution of the UIRs of the principal series are taken into account then any representation describing the interacting system is unitarily equivalent to the representation in BT form.

Consider first, in detail the case of two free particles in the non-relativistic approximation. We can write $\tilde{M} - m_1 - m_2$ in the form

$$M_{\mathrm{nr}} = \frac{k^2}{2m_{12}} + \frac{\iota}{R}\left(\boldsymbol{k}\frac{\partial}{\partial \boldsymbol{k}} + \frac{3}{2}\right) \tag{38}$$

where $m_{12} = m_1 m_2/(m_1 + m_2)$ is the reduced mass of particles 1 and 2. In coordinate representation this operator has the form

$$M_{\mathrm{nr}} = -\frac{\Delta}{2m_{12}} - \frac{\iota}{R}\left(\boldsymbol{r}\frac{\partial}{\partial \boldsymbol{r}} + \frac{3}{2}\right) \tag{39}$$

where $\Delta = (\partial/\partial \boldsymbol{r})^2$.

It is obvious that

$$M_{\mathrm{nr}} = \exp\left(\frac{\iota R k^2}{4m_{12}}\right) M_0 \exp\left(-\frac{\iota R k^2}{4m_{12}}\right) \tag{40}$$

where M_0 is given by equation (30). Therefore, we have the following statement.

Statement 3. *The operators M_0 and M_{nr} are unitarily equivalent. In particular, M_{nr} contains only the continuous spectrum occupying the interval* $(-\infty, \infty)$.

The Hilbert space of functions satisfying conditions (24) or (25) can be decomposed into the subspaces H_{lm} such that the elements of H_{lm} have the form

$$f(\boldsymbol{k}) = Y_{lm}(\boldsymbol{k}/k) f(k) \qquad \varphi(\boldsymbol{r}) = Y_{lm}(\boldsymbol{r}/r)\varphi(r). \tag{41}$$

Y_{lm} is the spherical function, $\varphi(r)$ satisfies condition (26) and $f(k)$ satisfies the analogous condition in momentum representation.

In this representation the eigenvalue problem for the operator M_{nr} in H_{lm} does not depend on l and m

$$\frac{k^2}{2m_{12}} f_\lambda(k) + \frac{\iota}{R}\left(k\frac{\mathrm{d}f_\lambda(k)}{k} + \frac{3}{2} f_\lambda(k)\right) = \lambda f_\lambda(k). \tag{42}$$

The solution of this equation is

$$f_\lambda(k) = \frac{1}{k}\left(\frac{R}{2\pi k}\right)^{1/2} \exp\left(\iota R\left(\frac{k^2}{4m_{12}} - \lambda \log k\right)\right). \tag{43}$$

In coordinate representation the eigenvalue problem for the operator M_{nr} in H_{lm} reads

$$-\frac{1}{2m_{12}r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\varphi_{\lambda l}(r)}{\mathrm{d}r}\right) + \frac{l(l+1)}{2m_{12}r^2}\varphi_{\lambda l}(r) - \frac{\iota}{R}\left(r\frac{\mathrm{d}\varphi_{\lambda l}(r)}{\mathrm{d}r} + \frac{3}{2}\varphi_{\lambda l}(r)\right) = \lambda\varphi_{\lambda l}(r). \tag{44}$$

The relation between the functions $f_\lambda(k)$ and $\varphi_{\lambda l}(r)$ is given by the radial Fourier transform

$$\varphi_{\lambda l}(r) = \frac{R^{1/2}}{\pi}(-\iota)^l \int_0^\infty j_l(kr) k^{1/2} \exp\left(\iota R\left(\frac{k^2}{4m_{12}} - \lambda \log k\right)\right) \mathrm{d}k \tag{45}$$

where

$$j_l(kr) = \left(\frac{\pi}{2kr}\right)^{1/2} J_{l+1/2}(kr) \tag{46}$$

is the spherical Bessel function.

The integral in equation (45) can be calculated analytically. We use γ^2 to denote $-\iota R/4m_{12}$. Then [20]

$$\varphi_{\lambda l}(r) = \left(\frac{R}{2\pi r}\right)^{1/2} \frac{(-\iota)^l}{2\Gamma(l+3/2)} \gamma^{\iota\lambda R-1} \left(\frac{r}{2\gamma}\right)^{l+1/2} \times \Gamma\left(\frac{l}{2}+\frac{3}{4}-\frac{\iota\lambda R}{2}\right) {}_1F_1\left(\frac{l}{2}+\frac{3}{4}-\frac{\iota\lambda R}{2}; l+\frac{3}{2}; \frac{r^2}{4\gamma^2}\right) \tag{47}$$

where Γ is the gamma function and ${}_1F_1$ is the hypergeometric function (in [20] a general case is considered which requires $\mathrm{Re}(\gamma^2) > 0$ but in our case it is also possible to use the result of [20] if $\mathrm{Re}(\gamma^2) = 0$).

As follows from equation (47), when $r \to 0$, the function $\varphi_{\lambda l}(r)$ is proportional to r^l. This fact is clear from analogy with conventional quantum mechanics. Indeed, by analogy with the standard investigation of radial Schrödinger equations, one can expect that at $r \to 0$ the first two terms in equation (44) are dominant. It is well known that the only regular solution of such conditions is proportional to r^l.

Let us now consider the asymptotic behaviour of $\varphi_{\lambda l}(r)$ when $r \to \infty$. It is convenient to use not equation (47) but the original integral in (45). Introducing the new integration variable $t = kr$ and using equation (46) one arrives at the following asymptotic result

$$\varphi_{\lambda l}(r) = \frac{(-\iota)^l}{r}\left(\frac{R}{2\pi r}\right)^{1/2} \exp(\iota\lambda R) \int_0^\infty J_{l+1/2}(t) \exp(-\iota R\lambda \log t)\, \mathrm{d}t. \tag{48}$$

Since [20]

$$\int_0^\infty J_{l+1/2}(t) \exp(-\iota R\lambda \log t)\, \mathrm{d}t = 2^{-\iota R\lambda}\Gamma\left(\frac{l}{2}+\frac{3}{4}-\frac{\iota\lambda R}{2}\right) \Big/ \Gamma\left(\frac{l}{2}+\frac{3}{4}+\frac{\iota\lambda R}{2}\right) \tag{49}$$

comparison of equations (44), (48) and (49) with equations (27) and (28) shows that at $r \to \infty$ one can neglect the first two terms of equation (44).

We can normalize functions $\varphi_{\lambda l}(r)$ as

$$\int_0^\infty \varphi_{\lambda l}(r)^* \varphi_{\lambda l}(r) r^2\, \mathrm{d}r = \delta(\lambda - \lambda') \tag{50}$$

and any function $\varphi(\boldsymbol{r})$ from internal Hilbert space can be written as

$$\varphi(\boldsymbol{r}) = \sum_{lm} \int_{-\infty}^{\infty} c_{lm}(\lambda) Y_{lm}(\boldsymbol{r}/r) \varphi_{\lambda l}(r)\, \mathrm{d}\lambda. \tag{51}$$

Now, we proceed to the case of interacting particles and consider the operator

$$\hat{M}_{\mathrm{nr}} = -\frac{\Delta}{2m_{12}} + V(r) - \frac{\iota}{R}\left(\boldsymbol{r}\frac{\partial}{\partial \boldsymbol{r}} + \frac{3}{2}\right). \tag{52}$$

The eigenvalue problem for this operator in H_{lm} has the form

$$-\frac{1}{2m_{12}r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\psi_{\lambda l}(r)}{\mathrm{d}r}\right) + \frac{l(l+1)}{2m_{12}r^2}\psi_{\lambda l}(r) + V(r)\psi_{\lambda l}(r) - \frac{\iota}{R}\left(r\frac{\mathrm{d}\psi_{\lambda l}(r)}{\mathrm{d}r} + \frac{3}{2}\psi_{\lambda l}(r)\right) = \lambda\psi_{\lambda l}(r). \tag{53}$$

Suppose that $V(r)r^2 \to 0$ when $r \to 0$. Then, the third term in equation (55) is negligible in comparison with the first two when $r \to 0$ (see, e.g., [21]). Therefore, the asymptotic behaviour of the function $\psi_{\lambda l}(r)$ at $r \to 0$ is the same as that of $\varphi_{\lambda l}(r)$, i.e. $\psi_{\lambda l}(r)$ is proportional to r^l. On the other hand, if $V(r) \to 0$ when $r \to \infty$ then the third term in equation (55) is negligible in comparison with the fourth when $r \to \infty$. Therefore, the asymptotic behaviour of $\psi_{\lambda l}(r)$ at $r \to \infty$ also coincides with that of $\varphi_{\lambda l}(r)$, i.e. $\psi_{\lambda l}(r)$ at $r \to \infty$ is proportional to the expression given by equation (48).

Since functions $\psi_{\lambda l}(r)$ and $\varphi_{\lambda l}(r)$ have the same asymptotic behaviour at $r \to 0$ and $r \to \infty$, we conclude that the operators M_{nr} and $\hat{M}_{\mathrm{nr}}$ have the same spectrum. Because normalization of the eigenfunctions belonging to the continuous spectrum is fully determined by the asymptotic behaviour of these functions at $r \to \infty$ [21], we can normalize functions $\psi_{\lambda l}(r)$ in the same way as $\varphi_{\lambda l}(r)$

$$\int_0^\infty \psi_{\lambda l}(r)^* \psi_{\lambda l}(r) r^2 \, \mathrm{d}r = \delta(\lambda - \lambda'). \tag{54}$$

Then, we can define the operator U as follows. If $\varphi(\boldsymbol{r})$ is given by equation (51) then

$$U\varphi(\boldsymbol{r}) = \sum_{lm} \int_{-\infty}^{\infty} c_{lm}(\lambda) Y_{lm}(\boldsymbol{r}/r) \psi_{\lambda l}(r) \, \mathrm{d}\lambda. \tag{55}$$

The operator U commutes with $\boldsymbol{S}$ by construction. As follows from equations (50) and (54), this operator is unitary and $\hat{M}_{\mathrm{nr}} = U M_{\mathrm{nr}} U^{-1}$. Therefore, we have statement 4.

Statement 4. *The operators $\hat{M}_{\mathrm{nr}}$ and M_{nr} are unitarily equivalent.*

In the non-relativistic approximation we have to consider functions $f(\boldsymbol{k})$ for which the important values of $\boldsymbol{k}$ are much smaller than the masses of the particles in question. Suppose also that R is very large and the functions satisfy the conditions of equation (34). In coordinate representation the actions of M_{nr} and $\hat{M}_{\mathrm{nr}}$ on such functions are practically indistinguishable from the actions of the operators

$$M_{\mathrm{nr}}^{\mathrm{G}} = -\frac{\Delta}{2m_{12}} \qquad \hat{M}_{\mathrm{nr}}^{\mathrm{G}} = -\frac{\Delta}{2m_{12}} + V(r) \tag{56}$$

respectively ('G' means 'Galilei'). However, the operators $M_{\mathrm{nr}}^{\mathrm{G}}$ and $\hat{M}_{\mathrm{nr}}^{\mathrm{G}}$ are not necessarily unitarily equivalent. For example, if $V(r) = -\mathrm{constant}/r$ and constant > 0 then the operator $M_{\mathrm{nr}}^{\mathrm{G}}$ only has a continuous spectrum in the interval $[0, \infty)$ while $\hat{M}_{\mathrm{nr}}^{\mathrm{G}}$ has also a discrete spectrum at some negative values of λ. Since $M_{\mathrm{nr}}^{\mathrm{G}}$ and $\hat{M}_{\mathrm{nr}}^{\mathrm{G}}$ in this case have different spectra, they cannot be unitarily equivalent.

The result formulated in statement 4 could be expected from physical considerations. Indeed, if $V(r)$ is not too singular when $r \to 0$, the phenomenon known as the 'fall onto the centre' (see, e.g., [21]) does not occur and the spectra of M_{nr} and $\hat{M}_{\mathrm{nr}}$ are defined by the asymptotic of the eigenfunctions of these operators at $r \to \infty$. Even if R is very large, there exist such values of r that the cosmological repulsion becomes dominant in comparison with kinetic and potential energies. Since this repulsion is present in both M_{nr} and $\hat{M}_{\mathrm{nr}}$, these operators have the same spectrum and, therefore, are unitarily equivalent.

Consider now a system of N particles with arbitrary velocities and suppose that the interactions between the particles can be described only in terms of the degrees of freedom characterizing the particles. The gravitational and electromagnetic interactions are not very singular in the sense that they fall off at infinity and do not lead to the fall onto the centre [21]. In the case of strong interactions the problem exists how to describe the interaction of coloured objects at large distances. Such an interaction is often modelled by attractive potentials which at infinity are proportional to r. In this case the force of attraction does not depend on r and, therefore, can be neglected in comparison with the cosmological repulsion. Therefore, it is natural to say that at infinity all realistic interactions are negligible in comparison with the cosmological repulsion (or by definition, the necessary condition for any interaction to be realistic is to be small in comparison with the cosmological repulsion when $r \to \infty$). For these reasons, in the case of realistic interactions, the asymptotic of the eigenfunctions of the interacting mass operator is again defined by the cosmological repulsion and is the same as the asymptotic of the eigenfunctions of the free mass operator discussed in the previous section. In turn, if the unitary operator realizing the equivalence of two mass operators commutes with S and the corresponding representations can be realized in BT form then they are unitarily equivalent. Therefore, we can formulate the following statement.

Statement 5. *In* SO(1, 4)*-invariant theory the interacting mass operator of the system of N particles described by the UIRs of the principal series is unitarily equivalent to the free mass operator.*

5. Discussion

A standard problem of perturbation theory for linear operators (see, e.g., [22]) is as follows. Let A and $\hat{A}$ be self-adjoint operators on Hilbert space. Suppose they have the same (absolutely) continuous spectrum (this is treated as the property of $\hat{A}$ to be in some sense a small perturbation of A). Suppose also for simplicity that A does not contain other points of the spectrum. Consider the wave operators

$$W_{\pm}(A, \hat{A}) = s - \lim_{t\to\pm\infty} \exp(\iota\hat{A}t)\exp(-\iota At) \tag{57}$$

where $s-\lim$ means the strong limit. If $\hat{A}$ has the same spectrum as A then there exist conditions when $W_{\pm}$ are unitary and

$$\hat{A} = W_{\pm}(A, \hat{A})AW_{\pm}(A, \hat{A})^{-1} \tag{58}$$

i.e. A and $\hat{A}$ are unitarily equivalent. If $\hat{A}$ also contains the discrete spectrum, operators A and $\hat{A}$ cannot be unitarily equivalent but there exist conditions when $W_{\pm}$ are isometric and the S-operator $S = W_{+}^{*}W_{-}$ is unitary.

As shown in the previous section, in the SO(1, 4)-invariant theory the interacting mass operator $\hat{M}$ of a many-particle system has the same spectrum as the free mass operator M and these operators are unitarily equivalent. The absence of bound states is a consequence of the fact that at large distances the cosmological repulsion is dominant. The choice of the unitary operator realizing the equivalence of $\hat{M}$ and M is obviously not unique. By analogy with the standard results of perturbation theory for linear operators one could expect that a possible choice is $W_{\pm}(M, \hat{M})$.

If R is very large and one considers only the subspace H^{P} of functions satisfying the conditions analogous to (34), (37) etc then the actions of $\hat{M}$ and M on these functions are practically indistinguishable from the actions of the corresponding operators in the Poincaré-invariant theory (obtained from M and $\hat{M}$ by neglecting cosmological repulsion). Therefore, in the SO(1, 4)-invariant theory there exist quasi-bound states: their lifetime is very large and goes to infinity when $R \to \infty$. It is clear that there exist conditions when the quasi-bound states are practically indistinguishable from bound ones. The finite lifetime is related to the fact that theoretically there exists a non-zero probability for quasi-bound particles to pass through the barrier separating the usual and cosmological distances. However, in practice this probability can be negligible.

It is important to note that the subspace H^{P}, where the results of Poincaré-invariant theory are valid, is only a small part of the full Hilbert space H. In particular, if $f \in H^{\mathrm{P}}$ then $\exp(\iota\hat{M}t)$ and $\exp(-\iota Mt)$ cannot belong to H^{P} if t is of order R/c. Therefore, it is natural to think that the standard scattering problem in H^{P} is meaningful only if $ct \ll R$.

In local quantum field theories the Hilbert space for the system under consideration is the Fock space describing a system of infinite numbers of particles. It is well known (see, e.g., [23]), that there exist serious mathematical difficulties in constructing well defined representation operators of the Poincaré group in such theories. The problem becomes much more complicated if the symmetry group is the group of motions of a curved spacetime (see, e.g., [24]). These considerations pose the problem whether in all realistic SO$(1, 4)$-invariant theories the interacting and free representation generators are unitarily equivalent. Another important problem is whether divergencies in standard local field theories are related to the fact that loop contributions to the S-matrix involve not only a set of states belonging to H^{P} but a much wider set of states for which it is not possible to neglect the effects of de Sitter invariance. In other words, these effects can play a role of regularizers for usual divergent expressions in the standard approach. For the case of SO$(2, 3)$ symmetry such a possibility has been considered in [25].

If the interacting and free representation generators are unitarily equivalent then the very notion of interaction is not fundamental. Indeed, in this case *there is no need to introduce interaction terms into the operators: we can always work with free operators and physics is defined by the subset of states important in the processes under consideration*. The subsets corresponding to different interactions are connected with each other by unitary transformations which necessarily depend on R. Indeed, if one reduces the free and interacting operators onto H^{P} and neglects cosmological repulsion, then, as noted before, the operators obtained in such a way are not unitarily equivalent in the general case.

In particular, one might think that for the situation corresponding to the pair of operators M_{nr} and $\hat{M}_{\mathrm{nr}}$ (see the previous section) the fundamental problem is not the choice of the potential $V(r)$ which should be added to M_{nr} but the choice of the unitary operator U realizing the unitary equivalence of M_{nr} and $\hat{M}_{\mathrm{nr}}$. In the framework of such an approach one might think that the fundamental quantities are those defining the operator U. In this case the gravitational constant, electric charges etc are functions of more fundamental quantities and R, in agreement with the famous Dirac hypothesis [26] about the dependence of physical constants on cosmological parameters. Moreover, if unitary equivalence of free and interacting representation operators for all realistic interactions has a place then they are fully defined by the present state vector of the Universe. This can be treated as a quantum analogue of the Mach principle according to which the local physical laws are defined by the distribution of masses in the Universe (the discussion of Mach's principle and its relation to general relativity and Dirac cosmology can be found in wide literature—see, e.g., [27] and references therein).

In summary, SO$(1, 4)$-invariant theories have rather unusual properties, in particular the mass operator has only continuous spectrum in the interval $(-\infty, \infty)$, bound states do not exist and the representations describing free and interacting systems are unitarily equivalent. At the same time SO$(1, 4)$ invariance does not contradict the existing experimental data and, therefore, the possibility exists that the SO$(1, 4)$ group is the symmetry group of nature. For these reasons the investigation of SO$(1, 4)$-invariant theories is of indubitable interest.

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